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NEUTRINO MIXING, CP/T VIOLATION AND TEXTURES IN FOUR-NEUTRINO MODELS

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Abstract

We examine the prospects for determining the neutrino mixing matrix and for observing CP and T violation in neutrino oscillations in four-neutrino models. We focus on a general class of four-neutrino models with two pairs of nearly degenerate mass eigenstates separated by approximately 1 eV, which can describe the solar, atmospheric and LSND neutrino data. We present a general parametrization of these models and discuss in detail the determination of the mixing parameters and the mass matrix texture from current and future neutrino data in the case where ν_e and ν_μ each mix primarily with one other neutrino. We find that measurable CP/T violation in long-baseline experiments, with amplitude at the level of the LSND signal, is possible given current experimental constraints. Also, additional oscillation effects in short- and long-baseline experiments may be measurable in many cases. We point out that, given separate scales for the mass-squared differences of the solar and atmospheric oscillations, observable CP/T violation effects in neutrino oscillations signals the existence of a sterile neutrino. We examine several textures of the neutrino mass matrix and determine which textures can have measurable CP/T violation in neutrino oscillations in long-baseline experiments. We also briefly discuss some possible origins of the neutrino mass terms in straightforward extensions of the Standard Model.

I. INTRODUCTION

Our view of the neutrino sector of the Standard Model has recently undergone a revolutionary change. Observations of solar neutrinos [1–5], atmospheric neutrinos [6–8], and accelerator neutrinos [9] all indicate deviations from their predicted values in the Standard Model with massless neutrinos. In each case the observation can be understood in terms of neutrino oscillations which in turn requires nondegenerate neutrino masses. The accelerator evidence for oscillations is least secure, with preliminary data from the KARMEN Collaboration [10] excluding some regions of oscillation parameters preferred by the LSND data [9]. Since the solar, atmospheric, and LSND neutrino experiments have different L/E (the ratio of oscillation distance to neutrino energy), different orders of magnitude of neutrino mass-squared differences δm^2 are required to properly describe all features of the data [11]. This need for three small but distinct mass-squared differences naturally leads to the consideration of more than three light neutrino flavors. Any additional light neutrino must be sterile, i.e., without Standard Model gauge interactions, to be consistent with the well-established LEP measurements of $Z \to \nu\bar{\nu}$ [12]. From quite general arguments it has been shown [13,14] that a neutrino spectrum with two pairs of nearly degenerate mass eigenstates, separated by a gap of order 1 eV, is required to satisfy all of the constraints from solar, atmospheric, accelerator, and reactor data.

Sterile neutrinos (which we we denote as ν_s) have long been considered as an option for neutrino oscillations [15,16]. More recently a number of models have been proposed that utilize one or more sterile neutrinos to describe the existing neutrino data [14,17–20] or to explain r-process nucleosynthesis [21]. However, if sterile neutrinos mix with active flavor neutrinos they may be stringently constrained by Big Bang nucleosynthesis (BBN). In standard BBN phenomenology, the mass-squared difference δm^2 and the mixing angle between a sterile and active neutrino must satisfy the bound

$$\delta m^2 \sin^2 2\theta < 10^{-7}\ \mathrm{eV}^2\,, \tag{1}$$

to avoid thermal overpopulation of the “extra”, sterile neutrino species [22]. The restriction in Eq. (1) would appear to rule out all sterile-active mixing except for small-angle MSW or vacuum mixing of solar neutrinos. However, some recent estimates of N_ν using higher inferred abundance of ^{4}He yield a considerably weaker bound than that given in Eq. (1) [23]. Thus BBN may still allow sizeable mixing between sterile and active neutrinos, so models with both small and large mixings with sterile neutrinos can be considered.

In this paper we examine the phenomenological consequences of four-neutrino models in which there are two pairs of neutrinos with nearly degenerate mass eigenstates separated by about 1 eV, where the mass separations within the pairs are several orders of magnitude smaller. We begin with a general parametrization of the four-neutrino mixing matrix, and review the current experimental constraints. We then discuss the simple situation where ν_e mixes dominantly with ν_s or ν_τ in solar neutrino oscillations and ν_μ mixes dominantly with a fourth neutrino (ν_τ or ν_s) in atmospheric neutrino oscillations. This situation, which we refer to as the dominant mixing scheme, has been shown to fit the existing data reasonably well. For dominant mixing we find that the neutrino mixing matrix can be effectively analyzed in terms of 2×2 blocks, where the diagonal blocks can be approximated by simple two-neutrino rotations and the off-diagonal blocks are small but non-vanishing. We then study

the relationship between the phenomenology of neutrino oscillations with CP violation and the texture of the neutrino mass matrix in models where the two lightest states are much lighter than the two heaviest states.

Since the neutrino mass matrix can in general be complex, and therefore lead to a mixing matrix with complex elements, CP violation can naturally arise in neutrino oscillations. The pertinent question is the size of the violation and how to observe it. We find that if CP violation exists, its size may be measurable, and has approximately the same amplitude as indicated by the LSND experiment. In many cases there are small amplitude $\nu_e \to \nu_\tau$ oscillations that may be measurable in either short- or long-baseline experiments. Furthermore, some also have small amplitude $\nu_\mu \to \nu_\tau$ oscillations in short-baseline experiments. We discuss how oscillation measurements in solar, atmospheric, short- and long-baseline neutrino experiments can, in some cases, determine all but one of the four-neutrino mixing matrix parameters accessible to oscillation measurements. We also discuss the minimal Higgs boson spectrum needed to obtain the different types of four-neutrino mass matrices, and their consequences for CP violation.

This paper is organized as follows. In Sec. II we present our parametrization for the four-neutrino mixing matrix and expressions for the oscillation probabilities in the case of two pairs of nearly degenerate mass eigenstates separated by about 1 eV. We discuss the ways in which CP violation, if it exists, may be observed, and the number of observable CP violation parameters. In Sec. III we summarize the current constraints on the four-neutrino mixing matrix and discuss in detail the implications of the dominant mixing scheme. We investigate the CP violation effects for several mass matrix textures in Sec. IV. In Sec. V we briefly discuss some of the consequences of neutrino mass for non-oscillation experiments, such as rare decays and charged lepton electric dipole moments, and we emphasize the importance in searching for these rare events, which can reveal new physics effects other than neutrino masses. In Sec. VI we summarize our results. Finally, in Appendix A we review the number of independent parameters in the mixing matrix of Majorana neutrinos, in Appendix B we discuss the modest extensions of the Standard Model Higgs sector that allow us to obtain the mass matrix textures, and in Appendix C we determine the neutrino mass spectrum and mixing matrix for a particular neutrino mass matrix with CP violation.

II. OSCILLATION PROBABILITIES

A. General Formalism

We work in the basis where the charged lepton mass matrix is diagonal. The most general neutrino mass matrix M is Majorana in nature, and may be diagonalized by a complex orthogonal transformation into a real diagonal matrix

$$M_D = U^T M U \,, \tag{2}$$

by a unitary matrix U, which is generally obtained from the Hermitian matrix $M^\dagger M = M^* M$ by $M_D^* M_D = U^\dagger M^\dagger M U$ [24]. Some general properties of Majorana neutrino mass matrices are discussed in more detail in Appendix A. For the four-neutrino case, labeling the flavor eigenstates by $\nu_x, \nu_e, \nu_\mu, \nu_y$ and the mass eigenstates by $\nu_0, \nu_1, \nu_2, \nu_3$, we may write

$$\begin{pmatrix} \nu_x \\ \nu_e \\ \nu_\mu \\ \nu_y \end{pmatrix} = U \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{3}$$

In this paper we will examine the two cases most often considered in the recent literature: one of ν_x and ν_y is ν_τ and the other is sterile (ν_s), or both ν_x and ν_y are sterile. Explicitly, the matrix M may be written in the flavor basis as

$$M = \begin{pmatrix} M_{xx} & M_{xe} & M_{x\mu} & M_{xy} \\ M_{xe} & M_{ee} & M_{e\mu} & M_{ey} \\ M_{x\mu} & M_{e\mu} & M_{\mu\mu} & M_{\mu y} \\ M_{xy} & M_{ey} & M_{\mu y} & M_{yy} \end{pmatrix} . \tag{4}$$

The 4×4 unitary matrix U may be parametrized by 6 rotation angles and 6 phases, and can be conveniently represented by [25]

$$U = R_{23} R_{13} R_{03} R_{12} R_{02} R_{01} , \tag{5}$$

where

$$R_{01} = \begin{pmatrix} c_{01} & s_{01}^* & 0 & 0 \\ -s_{01} & c_{01} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{6}$$

with

$$c_{jk} \equiv \cos\theta_{jk} , \qquad s_{jk} \equiv \sin\theta_{jk} e^{i\delta_{jk}} , \tag{7}$$

and the other R_{jk} are defined similarly for rotations in the j–k plane. The explicit form for the 4×4 unitary matrix is

$$U = \begin{pmatrix} c_{01}c_{02}c_{03} & c_{02}c_{03}s_{01}^* & c_{03}s_{02}^* & s_{03}^* \\ \\ \begin{matrix} -c_{01}c_{02}s_{03}s_{13}^* \\ -c_{01}c_{13}s_{02}s_{12}^* \\ -c_{12}c_{13}s_{01} \end{matrix} & \begin{matrix} -c_{02}s_{01}^*s_{03}s_{13}^* \\ -c_{13}s_{01}^*s_{02}s_{12}^* \\ +c_{01}c_{12}c_{13} \end{matrix} & \begin{matrix} -s_{02}^*s_{03}s_{13}^* \\ +c_{02}c_{13}s_{12}^* \\ \end{matrix} & \begin{matrix} c_{03}s_{13}^* \\ \\ \\ \end{matrix} \\ \\ \begin{matrix} -c_{01}c_{02}c_{13}s_{03}s_{23}^* \\ +c_{01}s_{02}s_{12}^*s_{13}s_{23}^* \\ -c_{01}c_{12}c_{23}s_{02} \\ +c_{12}s_{01}s_{13}s_{23}^* \\ +c_{23}s_{01}s_{12} \end{matrix} & \begin{matrix} -c_{02}c_{13}s_{01}^*s_{03}s_{23}^* \\ +s_{01}^*s_{02}s_{12}^*s_{13}s_{23}^* \\ -c_{12}c_{23}s_{01}^*s_{02} \\ -c_{01}c_{12}s_{13}s_{23}^* \\ -c_{01}c_{23}s_{12} \end{matrix} & \begin{matrix} -c_{13}s_{02}^*s_{03}s_{23}^* \\ -c_{02}s_{12}^*s_{13}s_{23}^* \\ +c_{02}c_{12}c_{23} \\ \\ \\ \end{matrix} & \begin{matrix} c_{03}c_{13}s_{23}^* \\ \\ \\ \\ \\ \end{matrix} \\ \\ \begin{matrix} -c_{01}c_{02}c_{13}c_{23}s_{03} \\ +c_{01}c_{23}s_{02}s_{12}^*s_{13} \\ +c_{01}c_{12}s_{02}s_{23} \\ +c_{12}c_{23}s_{01}s_{13} \\ -s_{01}s_{12}s_{23} \end{matrix} & \begin{matrix} -c_{02}c_{13}c_{23}s_{01}^*s_{03} \\ +c_{23}s_{01}^*s_{02}s_{12}^*s_{13} \\ +c_{12}s_{01}^*s_{02}s_{23} \\ -c_{01}c_{12}c_{23}s_{13} \\ +c_{01}s_{12}s_{23} \end{matrix} & \begin{matrix} -c_{13}c_{23}s_{02}^*s_{03} \\ -c_{02}c_{23}s_{12}^*s_{13} \\ -c_{02}c_{12}s_{23} \\ \\ \\ \end{matrix} & \begin{matrix} c_{03}c_{13}c_{23} \\ \\ \\ \\ \\ \end{matrix} \end{pmatrix} . \tag{8}$$

We will label the matrix elements of U by $U_{\alpha j}$, where Greek indices denote flavor eigenstate labels ($\alpha = x, e, \mu, y$) and Latin indices denote mass eigenstate labels ($j = 0, 1, 2, 3$). With the knowledge of the mass eigenvalues and mixing matrix elements, one can invert Eq. (2) to obtain the neutrino mass matrix elements

$$M_{\alpha\beta} = \sum_{j=0}^{3} U^*_{\alpha j} U^*_{\beta j} m_j \,. \tag{9}$$

The vacuum neutrino flavor oscillation probabilities, for an initially produced ν_α to a finally detected ν_β, can be written

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{j<k} \left[4 \ \mathrm{Re}(W^{jk}_{\alpha\beta}) \sin^2 \Delta_{kj} - 2 \ \mathrm{Im}(W^{jk}_{\alpha\beta}) \sin 2\Delta_{kj} \right] , \tag{10}$$

where

$$W^{jk}_{\alpha\beta} \equiv U_{\alpha j} U^*_{\alpha k} U^*_{\beta j} U_{\beta k} \,, \tag{11}$$

$$\Delta_{kj} \equiv \delta m^2_{kj} L/(4E) \,, \qquad \delta m^2_{kj} \equiv m^2_k - m^2_j \,, \tag{12}$$

L is the oscillation distance, and E is the neutrino energy. The quantities $W^{jk}_{\alpha\beta}$ [26], are related to the Jarlskog invariants [27]

$$J^{jk}_{\alpha\beta} \equiv \ \mathrm{Im}(W^{jk}_{\alpha\beta}) \,, \tag{13}$$

which satisfy the identity

$$J^{jk}_{\alpha\beta} = -J^{jk*}_{\alpha\beta} \,, \tag{14}$$

obtained by the interchange $U \leftrightarrow U^*$ in Eq. (11). Also,

$$J^{jk}_{\alpha\beta} = J^{kj}_{\beta\alpha} = -J^{jk}_{\beta\alpha} = -J^{kj}_{\alpha\beta} \,. \tag{15}$$

We can also define the real part of $W^{jk}_{\alpha\beta}$ as

$$Y^{jk}_{\alpha\beta} \equiv \mathrm{Re}(W^{jk}_{\alpha\beta}) \,, \tag{16}$$

which is invariant under interchange of the upper or lower indices:

$$Y^{jk}_{\alpha\beta} = Y^{kj}_{\beta\alpha} = Y^{jk}_{\beta\alpha} = Y^{kj}_{\alpha\beta} \,. \tag{17}$$

Another useful property of the $W^{jk}_{\alpha\beta}$ is that the sum over any of the indices reduces them to a real positive quantity, e.g.,

$$\sum_\beta W^{jk}_{\alpha\beta} = |U_{\alpha j}|^2 \delta_{jk} = \sum_\beta Y^{jk}_{\alpha\beta} \,, \tag{18}$$

$$\sum_\beta J^{jk}_{\alpha\beta} = 0 \,. \tag{19}$$

Equations (10), (14), (15) and (17) imply

$$P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)\,, \tag{20}$$

which is a statement of CPT invariance. Equation (10) and (15) imply that nonzero $J^{jk}_{\alpha\beta}$ can give CP or T violation

$$P(\nu_\alpha \to \nu_\beta) \neq P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\nu_\beta \to \nu_\alpha)\,. \tag{21}$$

From Eq. (21) we can define the CP-violation quantity

$$\Delta P_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)\,. \tag{22}$$

In four-neutrino oscillations there are only three independent $\Delta P_{\alpha\beta}$, and, correspondingly, only three of the six phases in U can be measured in neutrino oscillations (for a discussion, see Appendix A). Thus six angles and three phases can in principle be measured in neutrino oscillations, which is the same as in the Dirac neutrino case. Therefore, as far as neutrino oscillations are concerned, our results apply equally to Dirac neutrinos. The three remaining independent phases in U enter into the mass matrix elements and processes such as neutrinoless double beta decay.

B. Model with two nearly degenerate pairs of neutrinos

For a four-neutrino model to describe the solar, atmospheric, LSND data and also satisfy all other accelerator and reactor limits, it must have two pairs of nearly degenerate mass eigenstates [13,14]; e.g., $\delta m^2_{sun} \equiv \delta m^2_{01} \ll \delta m^2_{atm} \equiv \delta m^2_{32} \ll \delta m^2_{LSND} \equiv \delta m^2_{21}$. We will also assume without loss of generality that $0 < m_0, m_1 < m_2 < m_3$. An alternative scenario with the roles of δm^2_{01} and δm^2_{32} reversed gives the same results as far as oscillations are concerned, although the implications for the mass matrix, double beta decay and cosmology may differ; this alternate possibility will be briefly discussed in Sec. IV.E. Also note that if the solar oscillations are driven by the MSW effect [28], we must require $m_0 > m_1$; for vacuum oscillations, $m_0 < m_1$ is also possible.

Given this hierarchy of the δm^2, the oscillation probabilities for $\alpha \neq \beta$ may be written approximately as

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) \simeq\ & A^{\alpha\beta}_{LSND} \sin^2 \Delta_{LSND} + A^{\alpha\beta}_{atm} \sin^2 \Delta_{atm} + B^{\alpha\beta}_{atm} \sin 2\Delta_{atm} \\ & + A^{\alpha\beta}_{sun} \sin^2 \Delta_{sun} + B^{\alpha\beta}_{sun} \sin 2\Delta_{sun}\,, \qquad \alpha \neq \beta\,, \end{aligned} \tag{23}$$

and for the diagonal channels

$$P(\nu_\alpha \to \nu_\alpha) \simeq 1 - A^{\alpha\alpha}_{LSND} \sin^2 \Delta_{LSND} - A^{\alpha\alpha}_{atm} \sin^2 \Delta_{atm} - A^{\alpha\alpha}_{sun} \sin^2 \Delta_{sun}\,, \tag{24}$$

where $\Delta_{scale} \equiv \frac{1}{4}\delta m^2_{scale} L/E$, $A^{\alpha\beta}_{scale}$ is the usual CP conserving oscillation amplitude for $\nu_\alpha \to \nu_\beta$ at a given oscillation scale, $B^{\alpha\beta}_{scale}$ is the CP violation parameter at a given scale, and the scale label is *sun* for the solar neutrino scale, *atm* for atmospheric and long-baseline scales, and *LSND* for accelerator and short-baseline scales. Note that the CP-violating terms have a different dependence on L/E from the CP-conserving terms, which could in principle be distinguished by measurements at different L/E [29]. In Eqs. (23) and (24) we have used the approximation $\Delta_{31} \simeq \Delta_{30} \simeq \Delta_{21} \simeq \Delta_{20} \simeq \Delta_{LSND}$.

The oscillation amplitudes are given by

$$
\begin{aligned}
A^{\alpha\beta}_{LSND} &= 4|U_{\alpha 2}U^*_{\beta 2} + U_{\alpha 3}U^*_{\beta 3}|^2 = 4|U_{\alpha 0}U^*_{\beta 0} + U_{\alpha 1}U^*_{\beta 1}|^2 \,, \qquad \alpha \neq \beta \,, && (25)\\
A^{\alpha\alpha}_{LSND} &= 4(|U_{\alpha 2}|^2 + |U_{\alpha 3}|^2)(1 - |U_{\alpha 2}|^2 - |U_{\alpha 3}|^2) \,, \\
&= 4(|U_{\alpha 0}|^2 + |U_{\alpha 1}|^2)(1 - |U_{\alpha 0}|^2 - |U_{\alpha 1}|^2) \,, && (26)\\
A^{\alpha\beta}_{atm} &= -4 \ \mathrm{Re}(U_{\alpha 2}U^*_{\alpha 3}U^*_{\beta 2}U_{\beta 3}) \,, \qquad \alpha \neq \beta \,, && (27)\\
A^{\alpha\alpha}_{atm} &= 4|U_{\alpha 2}|^2|U_{\alpha 3}|^2 \,, && (28)\\
A^{\alpha\beta}_{sun} &= -4 \ \mathrm{Re}(U_{\alpha 0}U^*_{\alpha 1}U^*_{\beta 0}U_{\beta 1}) \,, \qquad \alpha \neq \beta \,, && (29)\\
A^{\alpha\alpha}_{sun} &= 4|U_{\alpha 0}|^2|U_{\alpha 1}|^2 \,, && (30)
\end{aligned}
$$

where the second equality in Eqs. (25) and (26) follows from the unitarity of U. We note that the form of the short-baseline oscillation amplitudes in Eqs. (25) and (26) are different from the cases of long-baseline, Eqs. (27) and (28), and solar, Eqs. (29) and (30). The difference is due to the fact that the short-baseline oscillations arise from four mass-squared differences ($\delta m^2_{20} \simeq \delta m^2_{30} \simeq \delta m^2_{21} \simeq \delta m^2_{31}$), while the long-baseline and solar oscillations arise from only one mass-squared difference (δm^2_{32} and δm^2_{01}, respectively). Probability conservation implies $A^{\alpha\alpha}_{scale} = \sum_{\beta\neq\alpha} A^{\alpha\beta}_{scale}$, which can easily be shown using the unitarity of U. The CP violation parameters are

$$
\begin{aligned}
B^{\alpha\beta}_{atm} &= -2 \ \mathrm{Im}(U_{\alpha 2}U^*_{\alpha 3}U^*_{\beta 2}U_{\beta 3}) \,, && (31)\\
B^{\alpha\beta}_{sun} &= 2 \ \mathrm{Im}(U_{\alpha 0}U^*_{\alpha 1}U^*_{\beta 0}U_{\beta 1}) \,. && (32)
\end{aligned}
$$

Since $B^{\alpha\alpha}_j = 0$, there is no CP violation in diagonal channels. The absence of $B^{\alpha\beta}_{LSND}$ in Eq. (23) shows that no observable CP violation is present for the leading oscillation [30], and CP violation may only be seen in experiments that probe non-leading scales, δm^2_{atm} or δm^2_{sun}.

For short-baseline experiments where only the leading oscillation argument Δ_{LSND} has had a chance to develop, the off-diagonal vacuum oscillation probabilities are

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &\simeq A^{\alpha\beta}_{LSND} \sin^2 \Delta_{LSND} \,, \qquad \alpha \neq \beta \,, && (33)\\
P(\nu_\alpha \to \nu_\alpha) &\simeq 1 - A^{\alpha\alpha}_{LSND} \sin^2 \Delta_{LSND} \,. && (34)
\end{aligned}
$$

For larger L/E (such as in atmospheric and long-baseline experiments), where the secondary oscillation has had time to develop, the vacuum oscillation probabilities are

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &\simeq \frac{1}{2} A^{\alpha\beta}_{LSND} + A^{\alpha\beta}_{atm} \sin^2 \Delta_{atm} + B^{\alpha\beta}_{atm} \sin 2\Delta_{atm} \,, \qquad \alpha \neq \beta \,, && (35)\\
P(\nu_\alpha \to \nu_\alpha) &\simeq 1 - \frac{1}{2} A^{\alpha\alpha}_{LSND} - A^{\alpha\alpha}_{atm} \sin^2 \Delta_{atm} \,. && (36)
\end{aligned}
$$

Here we have assumed that the leading oscillation has averaged, i.e., $\sin^2 \Delta_{LSND} \to \frac{1}{2}$. Finally, at the solar distance scale, when all oscillation effects have developed, the vacuum oscillation probabilities are

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &\simeq \frac{1}{2}(A^{\alpha\beta}_{LSND} + A^{\alpha\beta}_{atm}) + A^{\alpha\beta}_{sun} \sin^2 \Delta_{sun} + B^{\alpha\beta}_{sun} \sin 2\Delta_{sun} \,, \qquad \alpha \neq \beta \,, && (37)\\
P(\nu_\alpha \to \nu_\alpha) &\simeq 1 - \frac{1}{2}(A^{\alpha\alpha}_{LSND} + A^{\alpha\alpha}_{atm}) - A^{\alpha\alpha}_{sun} \sin^2 \Delta_{sun} \,, && (38)
\end{aligned}
$$

where $\sin^2 \Delta_{atm}$ has been averaged to $\frac{1}{2}$ and $\sin 2\Delta_{atm}$ has been averaged to 0; the CP violation at the δm^2_{atm} scale is washed out. CP violation is possible only in the off-diagonal channels, as noted before, and the solar neutrino ν_e survival measurement cannot be used to observed CP violation.

More generally, in any model for which the oscillation scales are well-separated and L/E is only large enough to probe the largest oscillation scale, CP-violating effects in neutrino oscillations will be unobservable (strictly speaking, they are suppressed to order $\Delta \ll 1$, where Δ is the oscillation argument for the second-largest oscillation scale) [30]. The CP-violating effects become observable when L/E is large enough to probe both the largest and second-largest oscillation scales. For a four-neutrino model with different oscillation scales to describe the solar, atmospheric and LSND data, this means that CP violation can only be detected in experiments with L/E at least as large as those found in atmospheric and long-baseline experiments.

As a corollary, in three-neutrino models with two oscillation scales describing only the solar and atmospheric data, CP violation has the potential to be observable only in experiments with L/E comparable to or larger than the solar experiments. However, a measurement of off-diagonal oscillation probabilities is required to see CP violation, and that is not possible in solar neutrino experiments. Hence, if the solar neutrino oscillation scale is well-established, the observation of a CP-violation effect in long-baseline experiments could imply that there are at least three separate neutrino mass-squared difference scales, and thus more than three neutrinos.

III. DETERMINING THE OSCILLATION PARAMETERS

In this section we first derive some general constraints imposed by current data on the neutrino mixing matrix for four-neutrino models favored by the data, i.e., with two pairs of nearly degenerate masses satisfying $\delta m^2_{01} \ll \delta m^2_{32} \ll \delta m^2_{21}$. We then determine the form of the mixing matrix under the assumption that ν_e and ν_x are mostly a mixture of ν_0 and ν_1 and provide the dominant solar neutrino oscillation, and ν_μ and ν_y are mostly a mixture of ν_2 and ν_3 and provide the dominant atmospheric neutrino oscillation, which is the form of most explicit models in the literature. Then we discuss the measurements needed to determine the parameters in the neutrino mixing matrix. The results of this section apply equally to the case where $m_2 < m_3 < m_0, m_1$. The more general case where ν_e and ν_μ have large mixing with more than one other neutrino is briefly discussed in Sec. IV.E.

A. Solar $\nu_e \to \nu_x$ and atmospheric $\nu_\mu \to \nu_y$

The flavor eigenstates are related to the mass eigenstates by Eq. (3). Using the formulae in Sec. II, the amplitudes for short-baseline oscillation (such as LSND, reactors, and other past accelerator oscillation searches) are

$$A^{e\mu}_{LSND} = 4|U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3}|^2 = 4|U_{e0}U^*_{\mu 0} + U_{e1}U^*_{\mu 1}|^2 \,, \tag{39}$$

$$\begin{aligned} A^{\mu\mu}_{LSND} &= 4(|U_{\mu 2}|^2 + |U_{\mu 3}|^2)(1 - |U_{\mu 2}|^2 - |U_{\mu 3}|^2) \\ &= 4(|U_{\mu 0}|^2 + |U_{\mu 1}|^2)(1 - |U_{\mu 0}|^2 - |U_{\mu 1}|^2) \,, \end{aligned} \tag{40}$$

$$
\begin{aligned}
A^{ee}_{LSND} &= 4(|U_{e2}|^2 + |U_{e3}|^2)(1 - |U_{e2}|^2 - |U_{e3}|^2) \\
&= 4(|U_{e0}|^2 + |U_{e1}|^2)(1 - |U_{e0}|^2 - |U_{e1}|^2)\,,
\end{aligned}
\tag{41}
$$

where the second equalities in each case result from the unitarity of U. For atmospheric and long-baseline oscillation, the amplitudes are

$$A^{\mu\mu}_{atm} = 4|U_{\mu 2}|^2|U_{\mu 3}|^2\,, \tag{42}$$

$$A^{e\mu}_{atm} = -4\ \mathrm{Re}(U_{e2}U^*_{e3}U^*_{\mu 2}U_{\mu 3})\,, \tag{43}$$

$$B^{e\mu}_{atm} = -2\ \mathrm{Im}(U_{e2}U^*_{e3}U^*_{\mu 2}U_{\mu 3})\,, \tag{44}$$

$$A^{ey}_{atm} = -4\ \mathrm{Re}(U_{e2}U^*_{e3}U^*_{y2}U_{y3})\,, \tag{45}$$

$$B^{ey}_{atm} = -2\ \mathrm{Im}(U_{e2}U^*_{e3}U^*_{y2}U_{y3})\,, \tag{46}$$

$$A^{\mu y}_{atm} = -4\ \mathrm{Re}(U_{\mu 2}U^*_{\mu 3}U^*_{y2}U_{y3})\,, \tag{47}$$

$$B^{\mu y}_{atm} = -2\ \mathrm{Im}(U_{\mu 2}U^*_{\mu 3}U^*_{y2}U_{y3})\,, \tag{48}$$

and in solar experiments

$$A^{ee}_{sun} = 4|U_{e0}|^2|U_{e1}|^2\,. \tag{49}$$

The atmospheric neutrino experiments favor large mixing of ν_μ at the atmospheric scale [14,31]

$$A^{\mu\mu}_{atm} > 0.8\,, \tag{50}$$

at 90% C.L.; then Eq. (42) and unitarity imply

$$|U_{\mu 2}|^2 + |U_{\mu 3}|^2 > 0.894\,, \qquad |U_{\mu 0}|^2 + |U_{\mu 1}|^2 < 0.106\,. \tag{51}$$

Also, the Bugey reactor constraint [32] gives

$$A^{ee}_{LSND} < 0.06\,, \tag{52}$$

over the indicated range for δm^2_{LSND}; then Eq. (41) implies

$$|U_{e2}|^2 + |U_{e3}|^2 < 0.016\,, \qquad |U_{e0}|^2 + |U_{e1}|^2 > 0.984\,. \tag{53}$$

Finally, the oscillation interpretation of the LSND results [9] gives

$$A^{e\mu}_{LSND} \equiv \epsilon^2\,, \tag{54}$$

where ϵ is experimentally constrained to the range [9]

$$0.05 < \epsilon < 0.20\,, \tag{55}$$

where the exact value depends on δm^2_{LSND}. Hence, the atmospheric and Bugey results imply that $|U_{e2}|$, $|U_{e3}|$, $|U_{\mu 0}|$, and $|U_{\mu 1}|$ are all approximately of order ϵ or smaller. We note that given these constraints the size of $A^{\mu\mu}_{LSND}$ must also be small, in agreement with the CDHS bound on ν_μ disappearance [33].

If we assume that it is only ν_y that mixes appreciably with ν_μ in the atmospheric experiments and only ν_x that mixes appreciably with ν_e in the solar experiments, then $|U_{x2}|$, $|U_{x3}|$, $|U_{y0}|$, and $|U_{y1}|$ must also be small, i.e., of order ϵ or less. The mixing matrix can therefore be seen to have the form

$$U = \left(\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array} \right) , \tag{56}$$

where the U_j $(j = 1, 2, 3, 4)$ are 2×2 matrices and the elements of U_2 and U_3 are at most of order ϵ in size. The matrix U_4 is approximately the 2×2 maximal mixing matrix (i.e., all elements have approximate magnitude $1/\sqrt{2}$) that describes atmospheric $\nu_\mu \to \nu_y$ oscillations, but U_1, which is approximately unitary by itself and which primarily describes the mixing in the solar neutrino sector, may have large (for vacuum oscillations) or small (for MSW oscillations) mixing.

The form of U in Eq. (56), with U_2 and $U_3 \sim \epsilon$, implies

$$|s_{02}|, |s_{03}|, |s_{12}|, |s_{13}| \sim \epsilon \, , \tag{57}$$

in the general parametrization of Eq. (8). After dropping terms second order in ϵ and smaller, U takes the form

$$U \simeq \begin{pmatrix} c_{01} & s_{01}^* & s_{02}^* & s_{03}^* \\ & & & \\ -s_{01} & c_{01} & s_{12}^* & s_{13}^* \\ & & & \\ \begin{array}{c} -c_{01}(s_{23}^* s_{03} + c_{23} s_{02}) \\ +s_{01}(s_{23}^* s_{13} + c_{23} s_{12}) \end{array} & \begin{array}{c} -s_{01}^*(s_{23}^* s_{03} + c_{23} s_{02}) \\ -c_{01}(s_{23}^* s_{13} + c_{23} s_{12}) \end{array} & c_{23} & s_{23}^* \\ & & & \\ \begin{array}{c} c_{01}(s_{23} s_{02} - c_{23} s_{03}) \\ -s_{01}(s_{23} s_{12} - c_{23} s_{13}) \end{array} & \begin{array}{c} s_{01}^*(s_{23} s_{02} - c_{23} s_{03}) \\ +c_{01}(s_{23} s_{12} - c_{23} s_{13}) \end{array} & -s_{23} & c_{23} \end{pmatrix} . \tag{58}$$

This matrix provides a general parametrization of the four-neutrino mixing in models where ν_e mixes primarily with ν_x at the solar mass-squared difference scale, and ν_μ mixes primarily with ν_y at the atmospheric mass-squared difference scale. Unitarity of U is satisfied to the order of ϵ.

Despite the fact that the expansion of the matrix elements of U in Eq. (58) is to the first order of ϵ, it still allows us, as shown below, to extract all of the interesting oscillation and CP violation effects, which are second order in ϵ. Care must be taken when the leading order result cancels, and sometimes it is helpful to use the unitarity of U to derive an alternate expression that gives the correct leading order answer, e.g., for $A^{\mu y}_{LSND}$, the first expression in Eq. (25) gives zero when the form of U in Eq. (58) is used, but the second expression gives a finite (and correct to leading order) result.

The off-diagonal oscillation amplitudes for the leading oscillation are

$$A^{e\mu}_{LSND} = 4|s_{12}c_{23} + s_{13}s_{23}^*|^2 \, , \tag{59}$$

$$A^{ey}_{LSND} = 4|s_{12}s_{23} - s_{13}c_{23}|^2 \, , \tag{60}$$

$$A^{\mu x}_{LSND} = 4|s_{02}c_{23} + s_{03}s_{23}^*|^2 \, , \tag{61}$$

$$A^{ex}_{LSND} = A^{\mu y}_{LSND} = O(\epsilon^4) \, . \tag{62}$$

For atmospheric and long-baseline experiments the oscillation amplitudes are

$$A^{e\mu}_{atm} = -A^{ey}_{atm} = -4c_{23}\mathrm{Re}(s^*_{12}s_{13}s^*_{23})\,, \tag{63}$$
$$B^{e\mu}_{atm} = -B^{ey}_{atm} = -2c_{23}\mathrm{Im}(s^*_{12}s_{13}s^*_{23})\,, \tag{64}$$
$$A^{ex}_{atm}, B^{ex}_{atm} = O(\epsilon^4)\,, \tag{65}$$
$$A^{\mu\mu}_{atm} \simeq A^{\mu y}_{atm} = \sin^2 2\theta_{23}\,, \tag{66}$$
$$A^{\mu x}_{atm} = -4c_{23}\mathrm{Re}(s^*_{02}s_{03}s^*_{23})\,, \tag{67}$$
$$B^{\mu x}_{atm} = 2c_{23}\mathrm{Im}(s^*_{02}s_{03}s^*_{23})\,, \tag{68}$$
$$B^{\mu y}_{atm} = -2c_{23}\mathrm{Im}[(s^*_{02}s_{03} + s^*_{12}s_{13})s^*_{23}]\,, \tag{69}$$

where θ_{23} is defined in Eq. (7). All oscillation amplitudes at the LSND scale and all oscillation amplitudes (including the CP violation amplitudes) other than $A^{\mu y}_{atm}$ at the atmospheric scale are at most of order ϵ^2. Since $A^{\mu y}_{atm}$ is large, $B^{\mu y}_{atm}$, which is of order ϵ^2, may be hard to measure since it involves taking the difference of two nearly equal large numbers. At the solar scale, we have

$$A^{ee}_{sun} \simeq A^{ex}_{sun} = \sin^2 2\theta_{01}\,. \tag{70}$$

There are 6 mixing angles and 3 independent phases that in principle may be measured in neutrino oscillations. In all, there are eight independent parameters involved in the observables in Eqs. (59)–(70), which are the six mixing angles θ_{01}, θ_{02}, θ_{03}, θ_{12}, θ_{13}, θ_{23}, and the two phases

$$\phi_0 \equiv \delta_{03} - \delta_{02} - \delta_{23}\,, \tag{71}$$
$$\phi_1 \equiv \delta_{13} - \delta_{12} - \delta_{23}\,. \tag{72}$$

These eight parameters could in principle be determined by measurements of the eight observables A^{ee}_{sun}, $A^{\mu\mu}_{atm}$, $A^{e\mu}_{LSND}$, $A^{e\mu}_{atm}$, $B^{e\mu}_{atm}$, $A^{\mu x}_{LSND}$, $A^{\mu x}_{atm}$, and $B^{\mu x}_{atm}$. Therefore if $\nu_x = \nu_\tau$, then all eight of these parameters could in principle be determined from the solar, atmospheric, short- and long-baseline experiments. This emphasizes the need for both short- and long-baseline measurements of all active oscillation channels, since the oscillation amplitudes involve different combinations of the parameters at short and long baselines. If ν_x is sterile, the three parameters θ_{02}, θ_{03} and ϕ_0 might be difficult to determine since they would involve the disappearance $\nu_\mu \to \nu_s$ that is at most of order ϵ^2 in magnitude. If $\nu_y = \nu_\tau$, the additional observables A^{ey}_{LSND}, A^{ey}_{atm}, and B^{ey}_{atm} can provide a consistency check on the parameters θ_{12}, θ_{13}, and ϕ_1.

We note that from the above results that many of the CP-violating amplitudes can be the same order of magnitude as the corresponding CP-conserving amplitudes, and hence potentially observable in high-statistics long-baseline experiments. The CP violation parameters $B^{e\mu}_{atm}$ and $B^{\mu x}_{atm}$ could be determined in vacuum by measuring probability differences $\Delta\bar{P}_{e\mu}$ and $\Delta\bar{P}_{\mu x}$, where

$$\Delta\bar{P}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)\,, \tag{73}$$

in long-baseline experiments, or the probability differences $\Delta P_{e\mu}$ and $\Delta P_{\mu x}$, where $\Delta P_{\alpha\beta}$, defined in Eq. (22), measures explicit T-violation. In a vacuum,

$$\Delta P_{\alpha\beta} = \Delta \bar{P}_{\alpha\beta} = 2B^{\alpha\beta}_{atm} \sin 2\Delta_{atm} \,. \tag{74}$$

Alternatively, one could measure CP asymmetries

$$\mathcal{A}^{CP}_{\alpha\beta} = \frac{P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)}{P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)} \,, \tag{75}$$

or T asymmetries

$$\mathcal{A}^{T}_{\alpha\beta} = \frac{P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)}{P(\nu_\alpha \to \nu_\beta) + P(\nu_\beta \to \nu_\alpha)} \,. \tag{76}$$

In vacuum, CPT invariance insures that $\Delta P_{\alpha\beta} = \Delta \bar{P}_{\alpha\beta}$ and $\mathcal{A}^{T}_{\alpha\beta} = \mathcal{A}^{CP}_{\alpha\beta}$. However, matter effects could induce a nonzero $\Delta \bar{P}_{\alpha\beta}$ or $\mathcal{A}^{CP}_{\alpha\beta}$ even in the absence of CP violation [18,34]. Since the matter effects in long-baseline experiments for $P(\nu_\alpha \to \nu_\beta)$ and $P(\nu_\beta \to \nu_\alpha)$ are the same, the quantities $\Delta P_{e\mu}$ and $\mathcal{A}^{T}_{e\mu}$, which can only be nonzero if there is explicit CP or T violation, may be preferable [18].

There remains a third independent phase that could have consequences for neutrino oscillations, but will in practice be difficult to measure. This phase, which could be taken as δ_{01} defined in Eq. (5), could be determined from CP violation in $\nu_e \leftrightarrow \nu_\mu$ or ν_y at the δm^2_{sun} scale, but this effect would require the measurement of an off-diagonal channel at the solar scale. Therefore it appears that a complete determination of the four-neutrino mixing matrix is not possible with conventional oscillation experiments. Table I lists all parameters that appear to be accessible to observation together with the principal observables that determine these parameters.

B. More general mixing scenarios

In general, both solar ν_e and atmospheric ν_μ could oscillate into mixtures of ν_x and ν_y. In this event θ_{02} and θ_{03} in Eq. (8) are not necessarily small. If one of ν_x and ν_y is the tau neutrino and the other sterile, there are several possible ways that the existence of such mixing could be determined [35]. Also, vacuum CP-violation effects involving ν_e will still be no larger than order ϵ^2 (due to the smallness of U_{e2} and U_{e3}), but there are potentially large CP-violation effects in long-baseline ν_μ-ν_y oscillations (as large as allowed by the unitarity of U) [18].

IV. CP VIOLATION AND NEUTRINO MASS TEXTURES

In this section we study the relationship between the neutrino mass texture and the possibility for observable CP violation (and, equivalently, T violation) in neutrino oscillations in four-neutrino models. We will consider models where one of ν_x and ν_y is sterile and the other is ν_τ (such as in Refs. [14], [19], [20], and [36]), and also models where both are sterile [37], which are two possible extensions of the Standard Model neutrinos. Note that in all earlier studies the mass matrices were taken to be real and no CP violation was possible. In Appendix B we discuss straightforward extensions of the Standard Model for the two

cases and show explicitly how their neutrino mass matrices can arise. In Secs. IV.A–IV.D we assume that

$$m_0 \simeq m_1 \ll m_2 \simeq m_3 \simeq \sqrt{\delta m^2_{LSND}}\,, \tag{77}$$

i.e., the lighter pair of nearly-degenerate mass eigenstates are much lighter than the heavier pair, also nearly degenerate, which is the structure of most explicit four-neutrino models in the literature. In Sec. IV.E we briefly discuss models with other mass hierarchies.

From Eqs. (9) and (58) the neutrino mass matrix elements involving ν_μ and ν_y are, to leading order in ϵ,

$$M_{\mu\mu} \simeq M^*_{yy} \simeq m(c^2_{23} + s^2_{23})\,, \tag{78}$$
$$M_{\mu y} \simeq im \sin 2\theta_{23} \sin\delta_{23}\,, \tag{79}$$
$$M_{e\mu} \simeq m(s_{12}c_{23} + s_{13}s_{23})\,, \tag{80}$$
$$M_{ey} \simeq m(s_{13}c_{23} - s_{12}s^*_{23})\,, \tag{81}$$
$$M_{x\mu} \simeq m(s_{02}c_{23} + s_{03}s_{23})\,, \tag{82}$$
$$M_{xy} \simeq m(s_{03}c_{23} - s_{02}s^*_{23})\,, \tag{83}$$

where we have used the relative sizes of the $U_{\alpha j}$ and mass eigenvalues, and the fact that to leading order in ϵ, $m_2 \simeq m_3 \equiv m$. Note that the s_{jk}, defined in Eq. (7), may be complex. For the mass matrix elements M_{xx}, M_{xe}, and M_{ee}, all four terms in Eq. (9) are small and may be of similar size (the first two are suppressed by the small values of m_0 and m_1, the last two by mixing angles of size ϵ); their values depend on the exact structure in the solar sector, which we do not specify here. More precise solar neutrino measurements would help to determine their values.

The three phases which enter in the expressions for the mass matrix elements given above are δ_{23} and

$$\phi'_0 \equiv \delta_{03} - \delta_{02} + \delta_{23}\,, \tag{84}$$
$$\phi'_1 \equiv \delta_{13} - \delta_{12} + \delta_{23}\,. \tag{85}$$

Only the phases ϕ_0 and ϕ_1, which can be measured in oscillation experiments, and ϕ'_0, which appears in the expressions for the mass matrix elements, are independent. The two phases $\delta_{23} = (\phi'_0 - \phi_0)/2$ and $\phi'_1 = \phi'_0 + \phi_1 - \phi_0$ are linearly dependent.

Equations (78)–(83) may be used to examine the implications of specific textures of the neutrino mass matrix. In the following, we discuss several specific textures of the neutrino mass matrix which have been considered in the literature. Their CP effects are particularly noted.

A. $M_{e\mu} = 0$

Specific examples of this class of models are given in Refs. [14], [19], and [20], in which $\nu_x = \nu_s$, $\nu_y = \nu_\tau$, and the mass matrices are taken to be real. In these models the mass matrices were chosen to minimize the number of parameters needed to provide the appropriate phenomenology, and a nonzero $M_{e\mu}$ is not required. In Ref. [14] the case $\nu_x = \nu_\tau$ and $\nu_y = \nu_s$ was also considered.

Using Eq. (80), $M_{e\mu}=0$ implies $s_{12}c_{23}\simeq -s_{13}s_{23}$, which in turn leads to

$$A^{e\mu}_{LSND} \simeq 16|s_{13}|^2|s_{23}|^2\sin^2\delta_{23}\,, \tag{86}$$

$$A^{ey}_{LSND} \simeq 4|s_{13}|^2(1-\sin^2 2\theta_{23}\sin^2\delta_{23})/c^2_{23}\,, \tag{87}$$

$$A^{e\mu}_{atm} = -A^{ey}_{atm} \simeq 4|s_{13}|^2|s_{23}|^2\cos 2\delta_{23}\,, \tag{88}$$

$$B^{e\mu}_{atm} = -B^{ey}_{atm} \simeq -2|s_{13}|^2|s_{23}|^2\sin 2\delta_{23}\,, \tag{89}$$

and $\phi_0=-2\delta_{23}$. Observable oscillations at LSND requires $\theta_{23}\neq 0,\pi$ and $\delta_{23}\neq 0,\pi$. Also, $\nu_e\to\nu_y$ oscillations in short-baseline experiments, $\nu_e\to\nu_\mu$ and $\nu_e\to\nu_y$ oscillations in long-baseline experiments, and CP violation in long-baseline experiments are possible, although not required, in this scenario. Finally, we have

$$|M_{\mu\mu}| \simeq |M_{yy}| \simeq m\sqrt{1-\sin^2 2\theta_{23}\sin^2\delta_{23}}\,, \tag{90}$$

$$|M_{\mu y}| \simeq m\sin 2\theta_{23}\sin\delta_{23}\,. \tag{91}$$

As one example, if we take $\delta_{23}\to\frac{\pi}{2}$ and $\theta_{23}\to\frac{\pi}{4}$, we obtain the model of Ref. [14], in which there is maximal ν_μ-ν_y mixing and $|M_{\mu\mu}|\simeq|M_{yy}|\ll|M_{\mu y}|$. Furthermore $A^{ey}_{LSND}=B^{e\mu}_{atm}=-B^{ey}_{atm}=0$, so ν_e oscillates only to ν_μ in short-baseline experiments and there is no visible CP violation in long-baseline experiments.

If we allow $\theta_{23}\neq\frac{\pi}{4}$, we have a model equivalent to that of Ref. [19]; in this case $A^{ey}_{LSND}\neq 0$ and $B^{e\mu}_{atm}=-B^{ey}_{atm}=0$, so there can be $\nu_e\to\nu_y$ oscillations in short-baseline experiments but still no visible CP violation in long-baseline experiments.

In order to have CP violation in the present case, we must have $\delta_{23}\neq\frac{\pi}{2}$. Then there must be short-baseline $\nu_e\to\nu_y$ oscillations, although the existence of long-baseline $\nu_e\leftrightarrow\nu_\mu$ oscillations depends on the value of θ_{23}.

Finally, an interesting case to consider is maximal CP violation (maximal in the sense that it gives the largest CP-violation parameter for a given $|s_{13}|$ and $|s_{23}|$), which corresponds to $\delta_{23}=\frac{\pi}{4}$. If there is also maximal ν_μ-ν_y mixing ($\theta_{23}=\frac{\pi}{4}$), then the mass matrix in the ν_μ-ν_y sector is approximately (after appropriate changes of neutrino phase to make the diagonal elements real to leading order in ϵ)

$$\begin{pmatrix} M_{\mu\mu} & M_{\mu y}\\ M_{y\mu} & M_{yy}\end{pmatrix} \simeq \frac{m}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} + \frac{\delta m^2_{atm}}{4m}e^{i\pi/4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\,. \tag{92}$$

The measurables in short- and long-baseline experiments are then

$$A^{e\mu}_{LSND} = A^{ey}_{LSND} \simeq 4|s_{13}|^2\,, \tag{93}$$

$$A^{e\mu}_{atm} = -A^{ey}_{atm} \simeq 0\,, \tag{94}$$

$$B^{e\mu}_{atm} = -B^{ey}_{atm} \simeq -|s^2_{13}|\,, \tag{95}$$

i.e., ν_e oscillates equally into ν_μ and ν_y in short-baseline experiments and there are no additional contributions to the CP-conserving part of these oscillations in long-baseline experiments. The vacuum CP and T asymmetries are especially simple in this case,

$$\mathcal{A}^{CP}_{e\mu} = \mathcal{A}^{T}_{e\mu} = -\mathcal{A}^{T}_{ey} \simeq -\frac{1}{2}\sin 2\Delta_{atm}\,, \tag{96}$$

as the dependence on $|s_{13}|^2$ cancels in the ratio. The particular models discussed above are summarized in Table II.

B. $M_{ey} = 0$

An example of this class of models with $\nu_x = \nu_s$ and $\nu_y = \nu_\tau$ is given in Ref. [36], where a nonzero M_{ey} was not needed to provide the appropriate phenomenology. From Eq. (81), $M_{ey} = 0$ implies $s_{12}s_{23}^* \simeq s_{13}c_{23}$, which leads to

$$\begin{aligned}
A_{LSND}^{e\mu} &\simeq 4|s_{13}|^2(1-\sin^2 2\theta_{23}\sin^2\delta_{23})/|s_{23}|^2\,, && (97)\\
A_{LSND}^{ey} &\simeq 16|s_{13}|^2 c_{23}^2 \sin^2\delta_{23}\,, && (98)\\
A_{atm}^{e\mu} = -A_{atm}^{ey} &\simeq -4|s_{13}|^2 c_{23}^2 \cos 2\delta_{23}\,, && (99)\\
B_{atm}^{e\mu} = -B_{atm}^{ey} &\simeq 2|s_{13}|^2 c_{23}^2 \sin 2\delta_{23}\,, && (100)
\end{aligned}$$

and $\phi_0 = -2\delta_{23}$. The existence of oscillations in LSND implies that $\delta_{23} \neq \frac{\pi}{2}$ or $\theta_{23} \neq \frac{\pi}{4}$. As with the $M_{e\mu} = 0$ case, it is possible to have $\nu_e \to \nu_y$ oscillations in short-baseline experiments, $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_y$ oscillations in long-baseline experiments, and CP violation in long-baseline experiments. The approximate magnitudes of the mass matrix elements $M_{\mu\mu}$, M_{yy}, and $M_{\mu y}$ are the same as given in Eqs. (90) and (91).

The limit $\delta_{23} \to 0$ and $\theta_{23} \simeq \frac{\pi}{4}$ reproduces the model in Ref. [36], which has no $\nu_e \to \nu_y$ in short-baseline experiments and no visible CP violation in long-baseline experiments. CP violation can occur if $\delta_{23} \neq 0, \frac{\pi}{2}, \pi$, in which case there are $\nu_e \to \nu_y$ oscillations in short-baseline experiments and there may be $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_y$ oscillations in long-baseline experiments, depending on the value of θ_{23}. The $M_{ey} = 0$ model with maximal CP violation and maximal ν_μ-ν_y mixing ($\delta_{23} = \theta_{23} = \frac{\pi}{4}$) has the same features as the $M_{e\mu} = 0$ maximal CP violation case in Sec. IV.A, except that the vacuum CP and T asymmetries in Eq. (96) have the opposite sign. The particular $M_{ey} = 0$ cases discussed here are also summarized in Table II.

C. $M_{e\mu} \neq 0$ and $M_{ey} \neq 0$

In this more general case, barring fortuitous cancellations one would expect from Eqs. (59)–(68) that there are $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_y$ oscillations in short- and long-baseline experiments, and CP violation in $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_y$ oscillations in long-baseline experiments.

For this texture, $|M_{\mu\mu}|, |M_{yy}| \ll |M_{\mu y}|$ does not necessarily exclude visible CP violation, unlike the cases $M_{e\mu} = 0$ or $M_{ey} = 0$. As an example, the mass matrix

$$M = m \begin{pmatrix} \epsilon_1 & \epsilon_2 e^{i\phi_2} & 0 & 0 \\ \epsilon_2 e^{i\phi_2} & 0 & \epsilon_5 & \epsilon_3 e^{i\phi_3} \\ 0 & \epsilon_5 & \epsilon_4 & e^{i\phi_1} \\ 0 & \epsilon_3 e^{i\phi_3} & e^{i\phi_1} & \epsilon_6 \end{pmatrix}, \qquad (101)$$

which is an extension of the model introduced in Ref. [14], leads to CP violation of order ϵ^2 in $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_y$ oscillations in long-baseline experiments. This mass matrix differs from the one in Ref. [14] in that the $M_{e\mu}$ and $M_{\mu e}$ elements, denoted as ϵ_5, are not zero, the $M_{\mu\mu}$ and M_{yy} elements, denoted as ϵ_4 and ϵ_6, respectively, are not necessarily equal, and the CP-violating phases are not set to zero. The diagonal elements of the mass matrix can be

taken to be real. Because $M_{ee} = M_{x\mu} = M_{xy} = 0$ there are only three independent phases. The mass eigenvalues and approximate mixing matrix for the mass matrix in Eq. (101) are given in Appendix C.

The largest off-diagonal short-baseline oscillation amplitudes in this case are

$$A^{e\mu}_{LSND} \simeq 4\epsilon_3^2 \,, \tag{102}$$

$$A^{ey}_{LSND} \simeq 4\epsilon_5^2 \,. \tag{103}$$

Short-baseline ν_μ-ν_y oscillations are of order ϵ^4. The largest long-baseline oscillation probabilities are

$$A^{\mu y}_{atm} \simeq 1 \,, \tag{104}$$

$$A^{e\mu}_{atm} = -A^{ey}_{atm} \simeq \epsilon_5^2 - \epsilon_3^2 \,. \tag{105}$$

Short- and long-baseline oscillation amplitudes involving ν_x are of order ϵ^4 or smaller. The observable CP-violating amplitude is

$$B^{e\mu}_{atm} \simeq \epsilon_3\epsilon_5 \sin(\phi_3 + \delta_{23}) \,, \tag{106}$$

where (see Appendix C)

$$\tan\delta_{23} = -\frac{(\epsilon_4 - \epsilon_6)\sin\phi_1 + 2\epsilon_3\epsilon_5\sin\phi_3}{(\epsilon_4 + \epsilon_6)\cos\phi_1 + 2\epsilon_3\epsilon_5\cos\phi_3} \,. \tag{107}$$

Finally,

$$A^{ee}_{sun} = \sin^2\theta_{01} \,, \tag{108}$$

where

$$\tan\theta_{01} = \frac{\epsilon_2}{\epsilon_1 + 2\epsilon_3\epsilon_5}\sqrt{1 + \frac{4\epsilon_1\epsilon_3\epsilon_5(1 - \cos(\phi_1 + \phi_3 - 2\phi_2))}{(\epsilon_1 - 2\epsilon_3\epsilon_5)^2}} \,. \tag{109}$$

The phenomenology of these models is summarized in Table II.

D. $M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$

In the special class of models with two sterile and two active neutrinos, i.e., both ν_x and ν_y are sterile, there need not be Majorana mass terms for the two active neutrinos in order to obtain the proper phenomenology. As described in more detail in Appendix B, $M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$ requires only a minimal extension in the Higgs sector of the Standard Model, i.e., only $SU(2)$ singlet Higgs bosons need to be added. Examples of models with both ν_x and ν_y sterile are given in Ref. [37]. We will now show that CP violation effects in long-baseline experiments are suppressed in this class of models.

Models with $M_{e\mu} = 0$ have already been discussed in Sec. IV.A; here we add the additional constraint $M_{\mu\mu} = 0$. It was previously determined that $|M_{\mu\mu}|, |M_{yy}| \ll |M_{\mu y}|$ implied $\delta_{23} \simeq \frac{\pi}{2}$ and $\theta_{23} \simeq \frac{\pi}{4}$, so Eqs. (86)–(89) reduce to

$$A^{e\mu}_{LSND} \simeq 8|s_{13}|^2 \,, \tag{110}$$

$$A^{ey}_{LSND} \simeq 0 \,, \tag{111}$$

$$A^{e\mu}_{atm} = -A^{ey}_{atm} \simeq -2|s_{13}|^2 \,, \tag{112}$$

$$B^{e\mu}_{atm} = -B^{ey}_{atm} \simeq 0 \,. \tag{113}$$

Hence, there is no visible CP violation in long-baseline experiments in this case (strictly speaking CP violation is strongly suppressed, to order ϵ^4). Therefore in models with $M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$, the only phenomenological deviations from the Standard Model are CP-conserving neutrino oscillations (see Table II) and the presence of more than one neutral Higgs scalar.

E. Other mass hierarchies

There are other mass hierarchies possible which give the same oscillation phenomena as those discussed in Secs. IV.A–IV.D. One is $m_2 < m_3 \ll m_0, m_1 \simeq \sqrt{\delta m^2_{LSND}} \equiv m$, in which the solar oscillation occurs between the two upper mass eigenstates and the atmospheric oscillation between the two lower mass eigenstates. Assuming as before that ν_e mixes primarily with ν_x and ν_μ with ν_y (i.e., $s_{02}, s_{03}, s_{12}, s_{13} \sim \epsilon$), then

$$M_{e\mu} \simeq m(U^*_{e0}U^*_{\mu 0} + U^*_{e1}U^*_{\mu 1}) \,; \tag{114}$$

M_{ee}, M_{ex}, M_{xx}, M_{ey}, $M_{x\mu}$, and M_{xy} are given by similar expressions with appropriate changes of subscripts. However, for $M_{\mu\mu}$, $M_{\mu y}$, and M_{yy} none of the four terms in Eq. (9) are dominant. Since long-baseline $\nu_\alpha \to \nu_\beta$ oscillations involve δm^2_{32}, and hence the mixing matrix elements $U_{\alpha 2}$, $U_{\alpha 3}$, $U_{\beta 2}$, and $U_{\beta 3}$ (see Eqs. (42)–(48)), then a mass texture condition such as $M_{e\mu} = 0$, when applied to Eq. (114), does not tell us anything specific about long-baseline oscillations. It could, however, affect short-baseline oscillation amplitudes, which depend on the $U_{\alpha 0}$ and $U_{\alpha 1}$ (see Eqs. (39)–(41)). We do not pursue this possibility further here.

Another possible hierarchy is to have $m_0 \simeq m_1 < m_2 \simeq m_3$ where none of the masses are much smaller than the others; in this case, all masses would contribute to hot dark matter (an alternate possibility, $m_2 \simeq m_3 < m_0 \simeq m_1$ with none small, gives similar results). A model of this type has been discussed in Ref. [20]. In this case, again assuming $s_{02}, s_{03}, s_{12}, s_{13} \sim \epsilon$, we find

$$M_{ex} \simeq m_0(U^*_{e0}U^*_{x0} + U^*_{e1}U^*_{x1}) \,, \tag{115}$$

with similar expressions for M_{ee} and M_{xx}, and

$$M_{\mu y} \simeq m_3(U^*_{\mu 2}U^*_{y2} + U^*_{\mu 3}U^*_{y3}) \,, \tag{116}$$

with similar expressions for $M_{\mu\mu}$ and M_{yy}. However, for $M_{e\mu}$, M_{ey}, $M_{x\mu}$, and M_{xy}, none of the four terms in Eq. (9) are dominant, the expressions for the mass matrix elements are more complicated, and the implications of particular textures for long-baseline oscillations are not as easily determined. We also do not pursue this case further here.

V. OTHER PROBES OF NEUTRINO MASS

The presence of mass terms for neutrinos and, in particular, Majorana mass terms, opens up a variety of possibilities for phenomena that are not possible in the Standard Model. The neutrino mass, besides giving rise to mixing of neutrinos and the associated CP effect discussed in this paper, can lead to lepton flavor-changing charged currents analogous to those of the quark sector. Majorana mass can also give rise to lepton number violation processes. With these possibilities, widely searched-for phenomena such as $\mu \to e + \gamma$, $\mu \to e + \bar{e} + e$, μ-e conversion, and electric dipole moments for charge leptons, can occur. Unfortunately, all these processes [38] are proportional to $(m_\nu/M_W)^4$ or $((m_\nu/M_W)ln(m_\nu^2/M_W^2))^4$. Given that m_ν is of the order of 5 eV or less [39] these are no larger than 10^{-40} and 10^{-33}. The current upper bounds [40] are about 30 orders of magnitude larger than those theoretical predictions from the neutrino masses. Therefore they are unobservable. Since this conclusion depends only on the smallness of the neutrino masses, it is valid in general.

The Majorana mass term breaks lepton number conservation and can lead to neutrinoless double beta decay. The rate is governed by the magnitude of the effective ν_e mass

$$\langle m_{\nu_e} \rangle = |\sum_j U_{ej}^2 m_j| = |M_{ee}|\,, \tag{117}$$

i.e., the magnitude of the M_{ee} element in the Majorana neutrino mass matrix in Eq. (4). The current limit on $|M_{ee}|$ from neutrinoless double beta decays is about 0.5 eV [41]. Since

$$M_{ee} = s_{01}^{*2} m_0 + c_{01}^2 m_1 + s_{12}^{*2} m_2 + s_{13}^{*2} m_3\,, \tag{118}$$

there will be no visible neutrinoless double beta decay in models with $m_0, m_1 \ll m_2 < m_3 \simeq \sqrt{\delta m_{LSND}^2} \approx 1$ eV and $|s_{12}| \approx |s_{13}| \approx \epsilon$. In models where $m_2 < m_3 \ll m_0, m_1 \simeq \sqrt{\delta m_{LSND}^2}$, or if no neutrino masses are $\ll 1$ eV, neutrinoless double beta decay may provide a strong constraint.

Neutrino masses may also affect cosmology if $\sum_\nu m_\nu > 0.5$ eV [42]. This level of neutrino mass can easily be accommodated by a four-neutrino model with two pairs of nearly degenerate mass eigenstates separated by approximately 1 eV.

Because of the smallness of the neutrino masses, there are no other observable effects besides neutrino oscillations and possibly neutrinoless double beta decay and dark matter. However, the rare decays may still be observable if new physics occurs also in other sectors, such as anomalous gauge boson couplings or anomalous fermion-gauge boson interactions. Therefore, it is important to continue to search for them.

VI. SUMMARY

In this paper we have presented a general parametrization of the four-neutrino mixing matrix and discussed the oscillation phenomenology for the case of two nearly degenerate pairs of mass eigenstates separated from each other by approximately 1 eV, which is the mass spectrum indicated by current solar, atmospheric, reactor, and accelerator neutrino experiments. We analyzed in detail the case where ν_e mixes primarily with ν_x and ν_μ with ν_y, where one of ν_x and ν_y is ν_τ and the other is sterile, or both are sterile. We found in these

cases that the neutrino mixing matrix can be written in 2×2 block form with small off-diagonal blocks. By construction the mixing matrices have $\nu_e \to \nu_\mu$ oscillations in LSND and $\nu_\mu \to \nu_y$ oscillations in atmospheric experiments. We found that the following oscillations are also possible: $\nu_e \to \nu_y$ and $\nu_\mu \to \nu_x$ in short-baseline experiments, and $\nu_e \to \nu_\mu$, $\nu_e \to \nu_y$, and $\nu_\mu \to \nu_x$ (including CP-violation effects) in long-baseline experiments. We also found that solar, atmospheric, short- and long-baseline oscillation measurements can, in some cases, determine all but one of the four-neutrino mixing parameters. Finally, we examined the implications of some several specific mass textures, and found the conditions under which CP-violation effects are visible.

As pointed out in the Introduction, additional evidence is needed in order for the four-neutrino scenario to be on firm ground. Given that there must be separate mass-squared difference scales for the solar and atmospheric oscillations (as currently indicated by the data), there are in fact two ways to verify the existence of four light neutrinos: (i) confirmation of the LSND results, which could occur in the future mini-BOONE collaboration [43–45], or (ii) detection of vacuum (i.e., not matter-induced) CP or T violation in long-baseline experiments, which should be greatly suppressed in a three-neutrino scenario.

Once the existence of four neutrinos is established, the next task is to determine the neutrino mixing matrix parameters. We emphasize the significant potential for detecting new oscillation channels and CP violation in future high statistics short- and long-baseline oscillation experiments. Many experiments have been proposed and some will be online in the next few years [45,46]. In these experiments neutrino beams are produced at high energy accelerators and oscillations can be detected at distant underground detectors. They include the KEK-Kamiokande K2K Collaboration [47], the Fermilab-Soudan MINOS [48] and Emulsion Sandwich [49] collaborations, and CERN-Gran Sasso ICARUS, Super-ICARUS, AQUA-RICH, NICE, NOE and OPERA collaborations [50]. Experiments done at muon storage rings [46] may be especially important since they will have the ability to measure both $\nu_e \to \nu_\mu$ and/or ν_τ, and $\nu_\mu \to \nu_e$ and/or ν_τ, as well as the corresponding oscillation channels for antineutrinos. Furthermore, there may also be hitherto undiscovered oscillation effects in short-baseline oscillation experiments such as COSMOS [51] and TOSCA [52], which will search for $\nu_\mu \to \nu_\tau$ oscillations. To completely determine all accessible parameters in the four-neutrino mixing matrix requires searches at both short and long baselines. The amplitudes of various oscillation channels, including possible CP violation effects, will help further determine the texture of the four-neutrino mass matrix and offer a better understanding of neutrino physics as well as CP violation.

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APPENDIX A: GENERAL PROPERTIES OF $N \times N$ MAJORANA MASS AND MIXING MATRICES

Consider the general case of n neutrinos, in which n_R are right-handed (sterile) and n_L are left-handed (active), $n_R + n_L = n$. We can represent the n_R right-handed neutrinos by their left-handed conjugates and denote the collection of all the n independent left-handed neutrinos by a column vector ψ_L. Then either Dirac or Majorana neutrino mass terms can be written as

$$\bar{\psi}_R M \psi_L + h.c.\,, \tag{A1}$$

with $\bar{\psi}_R$ related to ψ_L by $\bar{\psi}_R = \psi_L^T C$, where $C = i\gamma^2\gamma^0$ is the charge conjugation matrix. The most general $n \times n$ mass matrix M is symmetric and complex, $M^\dagger = M^*$, characterized by $\frac{1}{2}n(n+1)$ magnitudes and the same number of phases. Since a given component field in ψ_L and $\bar{\psi}_R$ acquires the same phase factor under a change of phase, n of the phases in M may be absorbed into the definitions of the fields, leaving $\frac{1}{2}n(n+1)$ magnitudes and $\frac{1}{2}n(n-1)$ phases; we will often choose the convention that the diagonal elements of M are real and the off-diagonal elements complex.

As shown in Eq. (2), the mass matrix may be diagonalized by a unitary matrix U. The effects of U may be divided into two classes of phenomenology which give rise to violation of individual lepton number: neutrino oscillations and lepton charged currents. Since Z^0 interacts only with the left-handed neutrinos, neutrino counting in $Z \to \nu\bar{\nu}'$ at LEP is unchanged if all of the neutrinos are light, i.e., $(N_\nu)_{LEP} = n_L$, with $n_L = 3$ in the Standard Model. This can be demonstrated straightforwardly as follows. The neutral current Lagrangian can be written in terms of flavor eigenstates as $g\bar{\psi}_L\gamma^\mu K \psi_L Z^0_\mu$, where K is an $n \times n$ diagonal matrix with the first n_R elements being zero and the other n_L elements unity. If all neutrinos are very light, which is the case we are considering here, the number of neutrinos measured at LEP is simply $\mathrm{Tr}(KK^\dagger) = n_L$, assuming the couplings of the active neutrinos are universal. In terms of the mass eigenstates, the neutrino counting is unchanged: $\mathrm{Tr}(UK(UK)^\dagger) = \mathrm{Tr}(KK^\dagger) = n_L$.

In general an $n \times n$ unitary matrix such as U can be described by $\frac{1}{2}n(n-1)$ rotation angles and $\frac{1}{2}n(n+1)$ phases. In the charged lepton current, we are free to make phase transformations of the charged lepton fields, which removes n of the phases. Then the number of surviving measurable phases is $\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)$. This argument is not affected by the fact that the number of left-handed charged lepton fields, n_c, in the charged-current is less than n, as each of the first $n - n_c$ rows of U can be multiplied by a phase without altering the charged currents. Therefore, in general in this Majorana setting we can parametrize U by $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}n(n-1)$ phases.

In neutrino oscillations, however, only $\frac{1}{2}(n-1)(n-2)$ independent phases can in principle be measured, as we will now demonstrate. Note that the $W^{jk}_{\alpha\beta}$ in Eq. (11) are invariant when U is transformed from either the left or right side by a diagonal matrix which contains only phases, i.e.,

$$U_{\alpha j} \to e^{i\phi_\alpha} U_{\alpha j} e^{i\phi_j}\,. \tag{A2}$$

Then, *as far as neutrino oscillations are concerned*, we can eliminate $2n-1$ of the phases in U, so that there are effectively only

$$\frac{1}{2}n(n+1) - (2n-1) = \frac{1}{2}(n-1)(n-2)\,, \tag{A3}$$

independent phases that can be measured by neutrino oscillation experiments. Interestingly this is the same number of independent phases that may be determined in the CKM matrix for n generations of quarks. However, note that in the most general U there are $\frac{1}{2}n(n-1)$ phases, so that there are $\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-2) = (n-1)$ phases that cannot be determined from neutrino oscillations.

The phase counting of Eq. (A3) can also be confirmed by enumerating the number of independent CP-violating variables, e.g., the $\Delta P_{\alpha\beta}$ defined in Eq. (22), that can be measured. There are $\frac{1}{2}n(n-1)$ such differences, but from Eq. (15) $\Delta P_{\alpha\beta} = -\Delta P_{\beta\alpha}$ and from Eq. (19) $\sum_\beta \Delta P_{\alpha\beta} = 0$, so it follows that $n-1$ of the differences are not independent. Therefore there are only $\frac{1}{2}(n-1)(n-2)$ independent $\Delta P_{\alpha\beta}$, and only $\frac{1}{2}(n-1)(n-2)$ CP-violation parameters can be measured.

APPENDIX B: HIGGS BOSON ORIGINS OF NEUTRINO MASSES

The presence of masses for neutrinos is a definite signal of physics beyond the Standard Model. Particularly, with the three types of neutrino oscillations which indicate three δm^2 scales and require at least four neutrino mass values, a non-trivial extension of the Standard Model is necessary. In searching for hints of the extension, it is interesting to consider what simplest extensions of the Standard Model are possible and how natural (or unnatural) they are in their couplings schemes. In this appendix we discuss briefly the possible origins of the two types of mass matrices considered, i.e., models with one or two sterile neutrinos. Then both Dirac (active-sterile) and Majorana (active-active or sterile-sterile) neutrino mass terms are present. These masses can be obtained by suitable extensions of the Standard Model. We will only enlarge the lepton Yukawa sector and the Higgs sector to the extent required by the mass matrices of Sec. IV. Our goal is to assure that such mass matrices are possible by straightforward modifications of these two sectors, and to determine what new particles need to be added to the Standard Model spectrum. We do not attempt to construct the best case scenario, which can be done when more information on the neutrinos mass are available. For a more extensive discussion of possible origins of neutrino masses terms, see Ref. [53].

In the following we denote the right-handed sterile by ν_{sjR} and the corresponding left-handed conjugate by $\hat\nu_{sjL} = \bar\nu^T_{sjR}C$. We also denote the left-handed lepton SU(2) doublet by l_{kL} with the corresponding right-handed conjugate fields $\hat l_{kR} = -i\sigma_2 C\bar l^T_{kR}$, j and k are generation labels.

1. Models with two left-handed and two right-handed neutrinos

We first consider the class of models in which $\nu_x = \nu_{s1}$ and $\nu_y = \nu_{s2}$, where ν_{s1} and ν_{s2} are the right-handed sterile neutrinos that are associated with ν_e and ν_μ, respectively, which have been considered in Ref. [37]. We discuss two cases: (i) models where only the sterile neutrinos have nonzero Majorana mass terms, and (ii) models where both right-handed (sterile) and left-handed (ν_e and ν_μ) fields have Majorana mass terms. In the first case

$M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$, which is discussed in Sec. IV.D; in the second case, $M_{e\mu} \neq 0$, which can be realized in the models discussed in Secs. IV.B and IV.C.

The simplest extension of the Standard Model is case (i) above, which can be obtained by adding a singlet real scalar field ϕ to the Standard Model Higgs doublet Φ. The Majorana masses for the two sterile are due to their coupling to ϕ and are proportional to the vacuum expectation value (vev) of ϕ. The Φ provides the Dirac masses from Yukawa couplings involving both sterile and left-handed neutrinos. We denote the absolute value of the vev's of the Φ and ϕ fields as v and V respectively; v is the same as the Standard Model vev. The Yukawa couplings can be written as

$$\mathcal{L}_Y = \sum_{j,j'}^{2} G_{jj'} \phi \bar{\nu}_{sjR} \hat{\nu}_{sj'L} + \sum_{j,k} g_{jk} (\bar{\nu}_{sjR} \tilde{\Phi}^\dagger l_{kL} + \bar{\hat{l}}_{kR} \Phi \hat{\nu}_{sR}) + h.c.\,, \tag{B1}$$

where $G_{jj'}$ and g_{jk} are complex couplings, and $\tilde{\Phi} = i\sigma\Phi^*$. Since $\bar{\nu}_{sjR}\hat{\nu}_{sj'L} = \hat{\nu}_{sj'L}\bar{\nu}_{sjR}$ we have $G_{jj'} = G_{j'j}$. We also exhibit the symmetry of the Dirac coupling coefficients, $g_{jk} = g_{kj}$, because of the identity $\bar{\nu}_{sjR}\tilde{\Phi}^\dagger l_{kL} = \bar{\hat{l}}_{kR}\Phi\hat{\nu}_{sjR}$

For the Higgs potential, we take the simplified case that it has a Z_2 symmetry in ϕ, i.e., invariant under $\phi \to -\phi$. Then the Higgs potential contains only real coefficients:

$$\mathcal{L}_{\mathcal{H}} = -\mu_1^2 |\Phi|^2 - \mu_2^2 \phi^2 + \lambda |\Phi|^4 + \lambda_2 \phi^4 + \lambda_3 \phi^2 |\Phi|^2\,. \tag{B2}$$

After spontaneous symmetry breaking there are two massive neutral Higgs bosons. Their masses are set by v and V. The value of v is the same as in the Standard Model. The coefficients of the Yukawa couplings must be very small in order to give neutrino masses of the order of eV. In the case $V \gg v$, one of the Higgs boson is composed mostly of the neutral field of Φ with a mass proportional to v and the other composed of mostly the ϕ field with a mass proportional to V. There are no other changes to the Standard Model phenomenology.

One can also have a more complicated scenario of case (ii) in which the Majorana masses of the two left-handed neutrinos are non-vanishing. These types of models can be constructed by the approach discussed below in Appendix B.2.

2. Models with three left-handed and one right-handed neutrinos

Here we consider the cases (i) $\nu_x = \nu_s$ and $\nu_y = \nu_\tau$, or (ii) $\nu_x = \nu_\tau$ and $\nu_y = \nu_s$, where ν_s is a sterile neutrino. In the first case, mass terms are needed in the ν_μ-ν_τ sector to provide the large mixing of atmospheric neutrinos, while in the second case Majorana mass terms are needed to provide mixing of solar neutrinos. There are no constraints on which terms in the mass matrix may be nonzero, but in each case Majorana masses of the left-handed (active) neutrinos must exist. The phenomenology of some of these models is discussed in Secs. IV.A–IV.C.

Majorana mass terms between left-handed neutrinos can arise from the introduction of a Higgs triplet which has lepton number -2 [54]. The Majorana mass of the sterile neutrino again comes from a Higgs singlet as discussed above. We denote the triplet by Δ

$$\Delta = \frac{1}{\sqrt{2}}\vec{\tau}\cdot\vec{\delta} = \begin{pmatrix} \delta^+/\sqrt{2} & \delta^{++} \\ \delta^0 & -\delta^+/\sqrt{2} \end{pmatrix}, \tag{B3}$$

and the value of the vev of δ^0 by Λ. Since Λ contributes to the masses of the W and Z^0 bosons differently, it has to be small in comparison with the vacuum expectation of the Φ, say $\Lambda/v < 10^{-2}$, so that the bulk of the electroweak gauge boson masses come from the doublet. Then, this will not upset the good agreement achieved by the Standard Model prediction for the ρ parameter.

The Yukawa couplings can be written as

$$\mathcal{L}_Y = G_s\phi\bar{\nu}_{sR}\hat{\nu}_{sL} + \sum_k g_k(\bar{\nu}_{sR}\tilde{\Phi}^\dagger l_{kL} + \bar{\hat{l}}_{kL}\Phi\nu_{sR}) + \sum_{k,k'} h_{kk'}\bar{\hat{l}}_{kR}\Delta l_{k'L} + h.c.\,, \tag{B4}$$

where G_s, g_k and $h_{kk'}$, k and $k' = 1, 2, 3$, are complex couplings. The symmetry of the Dirac couplings is explicitly exhibited because of the identity $\bar{\nu}_{sR}\tilde{\Phi}^\dagger l_{kL} = \bar{\hat{l}}_{kL}\Phi\nu_{sR}$. $h_{kk'}$ is symmetric, $h_{kk'} = h_{k'k}$, because of the identity $\bar{\hat{l}}_{kR}\Delta l_{k'L} = \bar{\hat{l}}_{k'R}\Delta l_{kL}$. For the Higgs potential, we can again take the simplified case that ϕ is a real scalar field and the Higgs potential has the Z_2 symmetry in ϕ:

$$\begin{aligned}\mathcal{L}_H = & -\mu_1^2|\Phi|^2 - \mu_2^2\phi^2 - \mu_3^2 Tr(\Delta\Delta^\dagger) + \eta\Phi^\dagger\Delta\tilde{\Phi} + \eta^*\tilde{\Phi}^\dagger\Delta^\dagger\Phi + \lambda_1|\Phi|^4 + \lambda_2\phi^4 \\ & + \lambda_3(Tr(\Delta\Delta^\dagger))^2 + \lambda_4 Tr((\Delta\Delta^\dagger)^2) + \lambda_5 Tr(\Delta^2(\Delta^\dagger)^2) + \lambda_6 Tr\Delta^2 Tr((\Delta^\dagger)^2) \\ & + \xi_1\phi^2|\Phi|^2 + \xi_2\phi^2 Tr(\Delta\Delta^\dagger) + \xi_3\Phi^\dagger\Delta\Delta^\dagger\Phi + \xi_4\Phi^\dagger\Delta^\dagger\Delta\Phi + \xi_5|\Phi|^2 Tr(\Delta\Delta^\dagger)\,. \end{aligned} \tag{B5}$$

The following terms in the Higgs potential,

$$\eta\Phi^\dagger\Delta\tilde{\Phi} + \eta^*\tilde{\Phi}^\dagger\Delta^\dagger\Phi\,, \tag{B6}$$

are needed to break the global lepton number invariance in order to avoid the appearance of a Goldstone boson called Majoron [54], and η is the only coupling that potentially can be complex.

To obtain the neutrino mass matrix we can also make phase tranformations on the fermion fields ν_{sR} and l_{kL} to make G_s and the diagonal terms h_{kk} real. If CP is not broken spontaneously, which we assume to be the case here, the vacuum expectation values of all neutral fields can be made real by phase transformations on the Higgs fields Φ and Δ. After spontaneous symmetry breaking this choice of phases for the Yukawa couplings agrees with the convention of the neutrino mass matrix discussed in Sec. II.

With complex couplings, CP violation can generally occur in the Higgs sector. However, η is required to be real by the minimization of the Higgs potential. Hence explicit CP violation does not occur in this extended Higgs scenario. A more complicated Higgs potential can be chosen to allow complex couplings so that CP violation can be manifest in the Higgs sector. We will not elaborate on this possibility here.

After spontaneous symmetry breaking, the physical Higgs boson spectrum contains a doubly charged pair, a singly charged pair, and four neutrals. The Goldstone bosons are mostly from the Higgs doublet Φ. The masses of two of the neutral Higgs bosons are proportional to v. The masses of the remaining two neutral Higgs boson are similar to those of the case of Appendix B.1. Again the Majorana couplings of the sterile neutrino and the

Dirac coupling of the sterile to the left-handed neutrinos are small. However, the Majorana couplings among the left-handed leptons do not have to be small if Λ is chosen to be the order of the neutrino masses, i.e., eV [55]; this can be done without leading to any small Higgs boson masses. Then the coupling of the singly and doubly charged Higgs boson to the charged leptons are not small. The production of these particles in a future high energy linear collider or muon collider is possible if they are not too heavy.

In this type of model, since the constraints $M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$ do not apply, the CP-violating parameter $\mathrm{Im}(U_{e2}U^*_{e3}U^*_{\mu 2}U_{\mu 3})$ is no longer constrained to be approximately zero. The CP violation can be of order ϵ^2, which is the same order as the oscillation probabilities themselves, and hence measurable. Examples of this type of model include the maximal CP violation models characterized by Eq. (92) and the mass matrix in Eq. (101).

We note that to produce the required neutrino masses and the phenomenologically interesting mass spectrum and expectation values of the Higgs boson fields in both models dicussed in this section, new hierarchy problems are introduced [55]. In our view, such hierarchy problems do not necessarily argue against the models. However, it does argue that any model of this sort should be included in a larger, more natural, scheme. Note that although the hierarchy in the expectation values of the Higgs boson fields sometimes leads to a fine tuning of the parameters, the small vacuum expection value of Δ may be obtained in a natural way [56]. For example, if $\mu_3 \gg v^2, V^2, \eta^2 \gg \Lambda^2$, then the minimization of the Higgs potential leads to the relation $\Lambda \simeq -\eta v^2/\mu_3 \ll v$.

There is a growing literature on the generation of neutrino masses. An intriguing class of models are those that generate mass dynamically by higher order loop effects [57]. We refer the reader to Ref. [58] for recent and extensive analyses of this possibility for four neutrinos. There are also models that use lepton-number violating interactions in R-parity violating supersymmetry for the generation of Majorana mass [59].

APPENDIX C: AN EXAMPLE WITH CP VIOLATION

In this appendix we derive the masses and mixing matrix for the model described by the mass matrix in Eq. (101). In general a 4×4 Majorana mass matrix can have six independent phases (see Appendix A), but since three of the mass matrix elements are zero, there are only three independent phases in this case. We have chosen to make the diagonal elements of M real.

To achieve the proper neutrino phenomenology, we assume the following hierarchy

$$\epsilon_2 \ll \epsilon_1, \epsilon_4, \epsilon_6 \ll \epsilon_3, \epsilon_5 \ll 1\,. \tag{C1}$$

The mass-squared eigenvalues are approximately given by

$$m_0^2 \simeq \epsilon_1^2 m^2\,, \qquad m_1^2 \simeq 4\epsilon_3^2\epsilon_5^2 m^2\,, \qquad m_{2,3}^2 \simeq (1+\epsilon_3^2+\epsilon_5^2 \mp \epsilon_0^2)m^2\,, \tag{C2}$$

where

$$\epsilon_0^4 = 4\epsilon_3^2\epsilon_5^2 + (\epsilon_4-\epsilon_6)^2 + 4\epsilon_4\epsilon_6 c_1^2 + 4\epsilon_3\epsilon_5[\epsilon_4\cos(\phi_1-\phi_3) + \epsilon_6\cos(\phi_1+\phi_3)]\,, \tag{C3}$$

with $c_j \equiv \cos\phi_j$ and $s_j \equiv \sin\phi_j$. The eigenvalues are related to the physical mass-squared differences by

$$\delta m^2_{LSND} = m_2^2 - m_1^2 \simeq m^2\,, \tag{C4}$$

$$\delta m^2_{atm} = m_3^2 - m_2^2 \simeq 2\epsilon_0^2 m^2\,, \tag{C5}$$

$$\delta m^2_{sun} = m_0^2 - m_1^2 \simeq (\epsilon_1^2 - 4\epsilon_3^2\epsilon_5^2)m^2\,. \tag{C6}$$

The size of δm^2_{sun} in Eq. (C6) implied by the hierarchy of Eq. (C1) means that the solar neutrino oscillations are of the MSW type. Therefore in order to have the proper MSW enhancement in the sun we must have $m_0^2 > m_1^2$, which implies $|\epsilon_1| > |2\epsilon_3\epsilon_5|$.

The matrix U that diagonalizes M via Eq. (2) is given approximately by

$$U \simeq \begin{pmatrix} c_{01} & s_{01}e^{-i\delta_{01}} & 0 & 0 \\ -s_{01}e^{i\delta_{01}} & c_{01} & \frac{e^{i\phi_1}}{\sqrt{2}}(\epsilon_3 e^{-i\phi_3} - \epsilon_5 e^{i\delta_{23}}) & \frac{e^{i\phi_1}}{\sqrt{2}}(\epsilon_3 e^{-i(\phi_3+\delta_{23}} + \epsilon_5) \\ \epsilon_3 s_{01} e^{i(\delta_{01}+\phi_3-\phi_1)} & -\epsilon_3 c_{01} e^{i(\phi_3-\phi_1)} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{-i\delta_{23}} \\ \epsilon_5 s_{01} e^{i(\delta_{01}-\phi_1)} & -\epsilon_5 c_{01} e^{-i\phi_1} & -\frac{1}{\sqrt{2}}e^{i\delta_{23}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{C7}$$

where

$$\tan\delta_{23} = -\frac{(\epsilon_4 - \epsilon_6)\sin\phi_1 + 2\epsilon_3\epsilon_5 s_3}{(\epsilon_4 + \epsilon_6)\cos\phi_1 + 2\epsilon_3\epsilon_5 c_3}\,, \tag{C8}$$

$$\delta_{01} = \tan^{-1}\left(\frac{2\epsilon_3\epsilon_5 s_\alpha}{\epsilon_1 - 2\epsilon_3\epsilon_5 c_\alpha}\right) - \phi_2\,, \tag{C9}$$

with $c_\alpha \equiv \cos_\alpha$, $s_\alpha \equiv \sin_\alpha$, $\alpha \equiv \phi_3 - \phi_1 - 2\phi_2$, and

$$\tan\theta_{01} = \frac{\epsilon_2}{\epsilon_1 + 2\epsilon_3\epsilon_5}\sqrt{1 + \frac{4\epsilon_1\epsilon_3\epsilon_5(1-c_\alpha)}{(\epsilon_1 - 2\epsilon_3\epsilon_5)^2}}\,. \tag{C10}$$

We note that this U has the form of of Eq. (56). It also can be seen to have the form of Eq. (58) if we set

$$\theta_{02} = \theta_{03} = 0\,, \qquad \theta_{23} = \frac{\pi}{4}\,, \tag{C11}$$

and make the identifications

$$s_{13}e^{-i\delta_{13}} = \frac{1}{\sqrt{2}}(\epsilon_3 e^{-i\phi_3} - \epsilon_5 e^{i\delta_{23}})e^{i\phi_1}\,, \tag{C12}$$

$$s_{12}e^{-i\delta_{12}} = \frac{1}{\sqrt{2}}(\epsilon_3 e^{-i(\phi_3+\delta_{23})} + \epsilon_5)e^{i\phi_1}\,. \tag{C13}$$

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TABLES

TABLE I. Parameters in the four-neutrino mixing matrix and the primary observables used to determine them.

Parameter(s)	Primary Observable(s)
θ_{01}	A^{ee}_{sun}
θ_{23}	$A^{\mu y}_{atm}$
θ_{12},θ_{13},$\phi_1 \equiv \delta_{13} - \delta_{12} - \delta_{23}$	$A^{e\mu}_{LSND}$,$A^{e\mu}_{atm}$,$B^{e\mu}_{atm}$
θ_{02},θ_{03},$\phi_0 \equiv \delta_{03} - \delta_{02} - \delta_{23}$	$A^{\mu x}_{LSND}$,$A^{\mu x}_{atm}$,$B^{\mu x}_{atm}$

TABLE II. Summary of some particular four-neutrino models for $m_0, m_1 \ll m_2 < m_3$ and $s_{02}, s_{03}, s_{12}, s_{13} \sim \epsilon$. All models in the table have been constructed to have short-baseline $\nu_\mu \to \nu_e$ ocsillations in agreement with the LSND data and large-amplitude $\nu_\mu \to \nu_y$ oscillations in atmospheric and long-baseline experiments; they also all have negligible $\nu_e \to \nu_x$ and $\nu_\mu \to \nu_y$ oscillations in short-baseline experiments. The size of $\nu_\mu \to \nu_x$ oscillations and CP violation in long-baseline $\nu_\mu \to \nu_y$ oscillations depend on other model parameters. In all cases, one of ν_x and ν_y could be ν_τ and the other sterile, or both could be sterile.

Texture	δ_{23}	θ_{23}	CP-conserving short-baseline $\nu_e \to \nu_y$	CP-conserving long-baseline $\nu_e \to \nu_\mu$	CP-violating long-baseline $\nu_e \to \nu_\mu, \nu_y$	Reference
$M_{e\mu} = 0$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	No	Yes	No	Ref. [14]
	$\frac{\pi}{2}$	$\neq \frac{\pi}{4}$	Yes	Yes	No	Ref. [19]
	$\neq \frac{\pi}{2}$	any	Yes	Maybe	Yes	Sec. IV.A
	$\frac{\pi}{4}$	$\frac{\pi}{4}$	Yes	No	Maximal	Eq. (92)
$M_{ey} = 0$	0	$\frac{\pi}{4}$	Yes	No	No	Ref. [36]
	$\neq 0, \frac{\pi}{2}, \pi$	$\neq \frac{\pi}{2}$	Yes	Maybe	Yes	Sec. IV.B
	$\frac{\pi}{4}$	$\frac{\pi}{4}$	Yes	No	Maximal	Eq. (92)
$M_{e\mu}, M_{ey} \neq 0$	varies	$\frac{\pi}{4}$	Yes	Yes	Yes	Eq. (101)
$M_{ee} = M_{e\mu} = M_{\mu\mu} = 0$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	No	Yes	No	Sec. IV.D